

BRODY CURVES IN COMPLICATED SETS

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ABSTRACT. For a hyperbolic generalized Hénon mapping (in the sense of [3]), J^+ , the boundary of the set of non-escaping points, is known as a complicated set and also known to admit a foliation by biholomorphic images of \mathbb{C} (see [3], [8]). We prove the existence of a leaf, which is injective Brody in \mathbb{P}^2 , in the foliation of J^+ for certain Hénon mappings (for the definition of injective Brodyness, see Section 3).

1. INTRODUCTION

The Brody curve first appeared in Brody's proof in [5] that every compact non-Kobayashi hyperbolic manifold contains a non-trivial holomorphic image of \mathbb{C} . The non-trivial entire curve is called a Brody curve. (See Section 3.)

Also, in [10], Gromov considered infinite dimensional geometry and introduced the concept of mean dimension, a topological invariant. As a main example, he considered the space of Brody curves. Recently, the space of Brody curves has been much studied. In particular, for the projective spaces, see [6], [7], [12], [15], [16], [17] and [18]. However, it is not easy to find interesting examples (cf. [1], [2]) except trivial ones such as polynomial mappings of \mathbb{C} to \mathbb{P}^2 and something of that type. From the perspective of geometry, it is interesting to see non-trivial examples.

In this paper, we construct an interesting example of Brody curves using the dynamics of a certain generalized Hénon mapping.

A generalized Hénon mapping f is defined simply by a polynomial diffeomorphism of \mathbb{C}^2 of the form $f(z, w) = (p(z) - aw, z)$ where p is a monic polynomial of one complex variable and a is a non-zero constant. The polynomial diffeomorphisms of this class are particularly

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important since in [9], Friedland and Milnor classified polynomial diffeomorphisms of \mathbb{C}^2 and showed that the only dynamically interesting polynomial diffeomorphisms are the finite compositions of generalized Hénon mappings. So, many mathematicians have studied them (listing some, cf [3], [4], [8], [11]).

Based on the dynamics of f , \mathbb{C}^2 can be divided into two regions. Let

$$K^+ := \{(z, w) \in \mathbb{C}^2 : \exists c > 0 \text{ such that } \|f^n(z, w)\| < c, \forall n \in \mathbb{N}\}.$$

Also, let $J^+ := \partial K^+$ and $U^+ := \mathbb{C}^2 \setminus K^+$. Let $g^+ : \mathbb{C}^2 \rightarrow \mathbb{R}$ denote the Green function associated to f . Then $U^+ = \{g^+ > 0\}$ and $K^+ = \{g^+ = 0\}$.

In [1] and [2], the author considered a foliation structure for U^+ . More precisely, it was proved that the level set $\mathcal{L}_c := \{g^+ = c\}$ with $c > 0$ is foliated by biholomorphic images of \mathbb{C} and each leaf is dense in \mathcal{L}_c in [11]. In [1] and [2], the author further proved that every leaf is an injective Brody curve of \mathbb{P}^2 with respect to the Fubini-Study metric. (For definitions, see Section 3.)

In this paper, we consider a similar structural property for J^+ . In [3], [4] and [8], the foliation structure of J^+ was studied. In particular, in [3], Bedford and Smillie proved that J^+ admits a foliation \mathcal{F}^+ by biholomorphic images of \mathbb{C} for a hyperbolic generalized Hénon mapping f . Here, the main theorem of this paper is as follows:

Theorem 1.1. *Let $f(z, w) = (p(z) - aw, z)$ where p is a monic polynomial of one complex variable and a is a non-zero constant. Assume that f is hyperbolic (in the sense of [3]) and $|a| \leq 1$. Then, in the natural foliation \mathcal{F}^+ of J^+ , there exists a leaf which is an injective Brody curve of \mathbb{P}^2 with respect to the Fubini-Study metric.*

This is interesting in the sense that if we restrict Theorem 1.1 to the Hénon mappings in [8], then J^+ has fractional Hausdorff dimension and we can have the density property: the closure of the injective Brody curve leaf is equal to J^+ . In general, these properties may not be true. On the other hand, in the sense that the Fubini-Study metric of a Brody curve is bounded, Theorem 1.1 can be considered as having tame behavior. So, the curve in Theorem 1.1 is a not-too-complicated curve in a complicated set. From these perspectives, this curve is another non-trivial example of injective Brody curves in \mathbb{P}^2 .

Note that different from the case of [1] and [2], due to the recurrent behavior in J^+ , it is not expected that we can locate the final injective Brody curve. We just know the existence of an injective Brody curve leaf in J^+ .

The main ingredients for Theorem 1.1 are the hyperbolicity of f (in the sense of [3]) and flow-boxes of the foliation of J^+ and they are quite different from those for [1] and [2].

Notation. We use $\Delta(a, r)$ for the disc in \mathbb{C} centered at $a \in \mathbb{C}$ and of radius $r > 0$ and Δ for the standard unit disc in \mathbb{C} . We denote by $\|\cdot\|$ the standard Euclidean metric of \mathbb{C}^2 and by $ds(P, V)$ the standard Fubini-Study metric on \mathbb{P}^2 of $V \in T_P \mathbb{P}^2$ at $P \in \mathbb{P}^2$. For a holomorphic curve $\gamma : U \rightarrow \mathbb{P}^2$ and for $\theta' \in U$, $\|\gamma(\theta')\|_{FS, \theta'}$ denotes $ds(\gamma(\theta'), d\gamma|_{\theta=\theta'}(\frac{d}{d\theta}))$ where U is an open subset of \mathbb{C} .

2. PRELIMINARIES

In this section, we recall some basic properties about generalized Hénon mappings. A generalized Hénon mapping is a holomorphic polynomial automorphism $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by

$$f(z, w) = (p(z) - aw, z)$$

where $p(z)$ is a monic polynomial of one complex variable z with degree $d \geq 2$ and $a \neq 0$. Then, $f^{-1}(z, w) = (w, (p(w) - z)/a)$.

Let \mathbb{P}^2 be the 2-dimensional complex projective space and

$$I_+ := [0 : 1 : 0] \quad \text{and} \quad I_- := [1 : 0 : 0]$$

in the homogeneous coordinate system of \mathbb{P}^2 . Then, f has the natural extension to $\tilde{f} : \mathbb{P}^2 \setminus \{I_+\} \rightarrow \mathbb{P}^2 \setminus \{I_+\}$ by

$$\tilde{f}([z : w : t]) = \left[t^d p\left(\frac{z}{t}\right) - awt^{d-1} : zt^{d-1} : t^d \right].$$

Similarly, f^{-1} also has the natural extension to $\widetilde{f^{-1}} : \mathbb{P}^2 \setminus \{I_-\} \rightarrow \mathbb{P}^2 \setminus \{I_-\}$ by

$$\widetilde{f^{-1}}([z : w : t]) = \left[wt^{d-1} : \frac{1}{a}(t^d p\left(\frac{w}{t}\right) - zt^{d-1}) : t^d \right].$$

We recall the following notions and properties related to the dynamics of f as in [11]. Let

$$K^\pm = \{p \in \mathbb{C}^2 : \{f^{\pm n}(p)\} \text{ is a bounded sequence of } n\}.$$

Let $J^\pm = \partial K^\pm$, $K = K^+ \cap K^-$, $J = J^+ \cap J^-$ and $U^\pm = \mathbb{C}^2 \setminus K^\pm$. Then, it is known that K^\pm is closed and U^\pm is open in \mathbb{C}^2 .

Proposition 2.1 (See [14]). *K^\pm , U^\pm , I_\pm , and \tilde{f} satisfy the following properties:*

- (1) I_- and I_+ are the super-attracting fixed points of \widetilde{f} and \widetilde{f}^{-1} , respectively,
- (2) any compact subset of U^\pm uniformly converges to I_\mp , respectively,
- (3) $\widetilde{f}(\{t=0\} \setminus I_+) = I_-$ and $\widetilde{f}^{-1}(\{t=0\} \setminus I_-) = I_+$, and
- (4) $\overline{K^+} = K^+ \cup I_+$ and $\overline{K^-} = K^- \cup I_-$.

The following theorem describes the behavior of J^+ .

Theorem 2.2 (Theorem 1.3 in [2]). *There is no non-trivial holomorphic curve, which passes through I_+ , and is supported in $\overline{K^+} \subseteq \mathbb{P}^2$.*

We recall hyperbolicity for generalized Hénon mappings in [3] (see [13] and also [8]). If a generalized Hénon mapping f is hyperbolic, there are continuous subbundles E_u and E_s such that $T\mathbb{C}_J^2 = E^s \oplus E^u$, and $Df(E^s) = E^s$ and $Df(E^u) = E^u$, and there exists constants $c > 0$ and $0 < \lambda < 1$ such that

$$\begin{aligned} \|Df^n|_{E^s}\| &< c\lambda^n, & n \geq 0 & \text{ and} \\ \|Df^{-n}|_{E^u}\| &< c\lambda^n, & n \geq 0. \end{aligned}$$

The Stable Manifold Theorem and Theorem 5.4 in [3] imply that for every $x \in J$, there exists a leaf \mathcal{L}_x in \mathcal{F}^+ such that $x \in \mathcal{L}_x$ and $T_x\mathcal{L}_x = E_x^s$ where \mathcal{F}^+ is the natural foliation of J^+ in [3].

3. BRODY CURVES

In this section, we briefly introduce the concepts of the *Brody curve* and the *injective Brody curve*.

Definition 3.1 (Brody Curve). Let M be a compact complex manifold with a smooth metric ds_M . Let $\psi : \mathbb{C} \rightarrow M$ be a non-constant holomorphic map.

The map ψ is said to be *Brody* if $\sup_{\theta \in \mathbb{C}} ds_M(\psi(\theta), d\psi(\frac{\partial}{\partial \theta})) < C_M$ for some constant $C_M > 0$. We call the image $\psi(\mathbb{C})$ a *Brody curve* in M . The curve $\psi(\mathbb{C})$ is said to be *injective Brody* if the parametrization ψ is injective.

In the rest of the paper, we only consider the Brody curves in \mathbb{P}^2 with respect to the standard Fubini-Study metric of \mathbb{P}^2 .

Below, we consider some trivial examples. The proofs are all straightforward from computations and so, we omit them.

Proposition 3.2. *Let α be a complex constant and p, q polynomials of one complex variable z . Then, all curves of the form $[z : p(z) : 1]$ and of the form $[p(z) \exp(z) : q(z) \exp(\alpha z) : 1]$ are Brody.*

However, not all holomorphic curves from \mathbb{C} to \mathbb{P}^2 are Brody. The mapping $z \rightarrow [e^z : e^{iz^2} : 1]$ is not Brody. For the verification, simply take $z = bi$ for real b and let b to ∞ . Even if we require them to be injective, not all injective curves from \mathbb{C} to \mathbb{P}^2 are Brody. The following gives us some examples of injective but non-Brody curves.

Proposition 3.3. *The map $f_n : z \rightarrow (z, \exp(z^n))$ is not Brody in $\mathbb{C}^2 \subset \mathbb{P}^2$ for $n \geq 3$. In particular, not all holomorphic images of \mathbb{C} in \mathbb{P}^2 are Brody.*

We close this section by pointing out a property of parametrizations of injective Brody curves.

Proposition 3.4. *For an injective Brody curve \mathcal{C} in \mathbb{P}^2 , every parametrization of \mathcal{C} has uniformly bounded Fubini-Study metrics. In short, the injective Brodyness property does not depend on the choice of the parametrization.*

Proof. Let $\phi_1, \phi_2 : \mathbb{C} \rightarrow \mathcal{C}$ be two biholomorphic parametrizations of \mathcal{C} . The composition $\phi_2^{-1} \circ \phi_1 : \mathbb{C} \rightarrow \mathbb{C}$ is a holomorphic automorphism of \mathbb{C} . From a theorem of one complex variable, $\phi_2^{-1} \circ \phi_1(z) = \alpha z + \beta$ for constants $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$. \square

4. PROOF OF THEOREM 1.1

In this section, we prove the main theorem. Basically, we follow the Brody reparametrization lemma. We assume all the hypotheses in Theorem 1.1.

Proof of Proposition 1.1 We first define a family of analytic discs. From Corollary 6.13 in [3], periodic points are dense in J . Pick a periodic point $P \in J$ and say N its period. Let \mathcal{L}_P be a leaf in the foliation \mathcal{F}^+ of J^+ passing through P as discussed in Section 2. Fix an analytic disc $\psi : \Delta \rightarrow \mathcal{L}_P$ such that $\psi(0) = P$ and $\|\psi\|_{FS,0} > 0$. Then we consider a family of analytic discs as follows:

$$\varphi_n := f^{-Nn} \circ \psi : \Delta \rightarrow \mathcal{L}_P.$$

Then, since \mathcal{L}_P is a stable manifold of P , from the hyperbolicity of f , $\|\varphi_n\|_{FS,0} \rightarrow \infty$ as $n \rightarrow \infty$.

Now we apply the Brody reparametrization lemma. Note that φ_n 's are holomorphic in a slightly larger disc. Define $H_n : \Delta \rightarrow \mathbb{R}^+$ by $H_n(\theta) := \|\varphi_n\|_{FS,\theta}(1 - |\theta|^2)$. Then, there exists $\theta_n \in \Delta$ such that $H_n(\theta_n) = \max_{\theta \in \Delta} H_n(\theta)$. For each n , define a Möbius transformation $\mu_n(\zeta) := (\zeta + \theta_n)/(1 + \overline{\theta_n}\zeta)$ mapping 0 to θ_n . Let $g_n := \varphi_n \circ \mu_n$. Then

$$\|g_n\|_{FS,\zeta}(1 - |\zeta|^2) = \|\varphi_n\|_{FS,\theta}|\mu_n'(\zeta)|(1 - |\zeta|^2) = \|\varphi_n\|_{FS,\theta}(1 - |\theta|^2).$$

So, $\|g_n\|_{FS,\zeta} \leq \|g_n\|_{FS,0}/(1 - |\zeta|^2)$. Let $R_n = \|g_n\|_{FS,0}$ and define $k_n(\theta) = g_n(\theta/R_n)$. Then,

$$\|k_n\|_{FS,\theta} = \frac{\|g_n\|_{FS,\theta/R_n}}{R_n} \leq \frac{\|g_n\|_{FS,0}}{R_n(1 - |\theta/R_n|^2)} \leq 2,$$

on $\Delta(0, R_n/2)$. Note that $\|k_n\|_{FS,0} = 1$ and that from the hyperbolicity of f , we see that $R_n \rightarrow \infty$ as $n \rightarrow \infty$. Hence, from a normal family argument and the compactness of \mathbb{P}^2 , we can find a non-trivial holomorphic map $\Phi : \mathbb{C} \rightarrow \overline{J^+}$ such that $\|\Phi\|_{FS,0} = 1$. From Proposition 2.1, we have $\overline{K^+} = K^+ \cup I_+$. However, Theorem 2.2 implies that $\Phi(\mathbb{C}) \subset J^+$.

We prove that the Brody curve $\Phi(\mathbb{C})$ sits in a single leaf of the foliation of J^+ . Suppose the contrary. Then, there exists two points $\alpha, \beta \in \mathbb{C}$ such that $\Phi(\alpha)$ and $\Phi(\beta)$ live in two different leaves and the two α, β are sufficiently close so that some fraction of the complex curve $\Phi(\mathbb{C})$ connecting $\Phi(\alpha)$ and $\Phi(\beta)$ sits in a single flow-box of the foliation of J^+ . Let $\gamma \subset \Phi(\mathbb{C})$ denote the complex curve connecting $\Phi(\alpha)$ and $\Phi(\beta)$. Then, there exists a constant $\epsilon > 0$ such that for any plaque T in the flow-box, $\sup_{(z,w) \in \gamma} \text{dist}((z,w), T) > \epsilon$ where $\text{dist}(\cdot, \cdot)$ is with respect to the standard Euclidean distance of \mathbb{C}^2 . This is a contradiction to the local uniform convergence since the image of each reparametrized analytic disc sits inside a leaf of the foliation \mathcal{F}^+ of J^+ .

We show that Φ is one-to-one. Suppose to the contrary that Φ is not one-to-one. Then, there are $\alpha, \beta \in \mathbb{C}$ and $q \in \Phi(\mathbb{C})$ such that $\alpha \neq \beta$ and $\Phi(\alpha) = \Phi(\beta) = q$. Consider a sufficiently large $R_q > 1$ such that $\alpha, \beta \in \Delta(0, R_q)$. Let F be a compact set of \mathbb{C}^2 such that its interior contains J . Consider a finite covering of $J^+ \cap F$ consisting of flow-boxes of \mathcal{F}^+ . Note that since $|a| \leq 1$, Theorem 5.9 in [3] says that for any leaf \mathcal{L} in J^+ , there exists a point $x \in J$ such that \mathcal{L} is a stable manifold of the point x . Since $\Phi(\Delta(0, 2R_q))$ live in a single leaf, there exists sufficiently large $N_q \in \mathbb{N}$ such that the analytic disc $f^{N_q}(\Phi(\Delta(0, 2R_q)))$ sits inside a flow-box in the finite covering. Again, since the image of each reparametrized analytic disc sits inside a leaf of the foliation of J^+ , the injectivity of limit maps is implied by the local uniform convergence to Φ of a subsequence $\{k_{n_j}\}$ of injective maps, projection onto the base direction in the flow-box, and the Hurwicz theorem.

Since there is no proper biholomorphic image of \mathbb{C} inside \mathbb{C} , the leaf containing the injective Brody curve itself is an injective Brody curve. This proves our theorem. \square

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